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A STUDY OF NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER USING HAAR WAVELET METHOD

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ABSTRACT

In addition, work is being done to improve the numerical stability of wavelet-based approaches for stiff and ill-conditioned problems, the kinds of issues that often arise in practical settings. This study has the potential to influence many areas of science and technology, including signal processing, image analysis, fluid dynamics, quantum physics, and many more. This study adds to the development of computational tools capable of accurately and efficiently tackling difficult issues by enhancing numerical solutions for integral and differential equations using wavelet approach. In order to better handle the wide variety of integral and differential equations found in scientific and engineering fields, researchers are working to improve the capabilities of numerical solutions based on the wavelet approach. This effort aspires to pave the road for more precise and efficient simulations and analyses of complex systems by developing and implementing novel algorithms and improving computational approaches.

KEYWORDS: Numerical Solution, Differential Equations, Fractional Order, Haar Wavelet Method

INTRODUCTION

Collaborations between mathematicians, physicists, and engineers led to the development of fractional calculus. With this connection, ideas began to emerge that went beyond just constructing new transformations. Fractional calculus emerged soon after the establishment of classical calculus, around the end of the 17th century. Leibniz, Caputo, Hadamard, Fourier, Lioville, and Riemann are credited with conducting the first systematic research. Despite the fact that fractional calculus is a natural development of calculus, it has historically had very little impact on the field of physics. One theory to explain their lack of appeal is that different people have different notions of what fractional derivatives really are. Another challenge is that, due to their non-local nature, fractional derivatives don't seem to have a clear geometrical interpretation. In contrast, fractional calculus has seen explosive growth in both pure mathematics and scientific applications over the last several decades. This is due to the fact that it allows for a more accurate modeling of physical phenomena that depends not only on the current moment in time but also on its own past. Applications in current fields such as differential and integral equations, plasma physics, image and signal processing, fluid mechanics, viscoelasticity, mathematical biology, electrochemistry, finance, and the social sciences have driven the development of fractional calculus in recent decades. It's undeniable that fractional calculus has emerged as a promising new tool for solving a wide range of

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mathematical, scientific, and engineering challenges. We recommend the monographs for further reading on the subject of fractional calculus and its many applications.

Here, we provide the essential terminology and mathematical foundations of the theory of fractional calculus to establish our conclusions.

Definition 1 Operator of order fractional integration in the Riemann-Liouville framework $\alpha \geq 0$ to be defined as the following:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t > 0, \qquad (1)$$

where $\Gamma(.)$is the standard gamma function, and there are certain characteristics of the operator $J\alpha$ are listed below:

$$(i) \quad J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t), \qquad \alpha, \beta > 0$$

$$(ii) \quad J^{\alpha} J^{\beta} f(t) = J^{\beta} J^{\alpha} f(t), \qquad \alpha, \beta > 0$$

$$(iii) \quad J^{\alpha} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} t^{\alpha+\beta}, \qquad \beta > -1.$$

When attempting to use fractional differential equations to simulate events in the actual world, the Riemann-Liouville derivative has certain drawbacks. As a result, we will present a new fractional differential operator, D, that has been updated. α using the theory of viscoelasticity [29] that was presented by Caputo.

Definition 2. Fractional Caputo derivative of $D\alpha$ to be defined as the following:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{m}(\tau)}{(t-\tau)^{\alpha-m+1}}\, d\tau, \qquad (2)$$

where $m-1 < \alpha \leq m, m \in N$.

To get the required order of fractional derivative, the Caputo method first calculates the ordinary derivative, and then the fractional integral.

Caputo fractional derivative operator, like integer-order differentiation, is a linear operation.

$$D^{\alpha}(\gamma f(t) + \delta g(t)) = \gamma D^{\alpha} f(t) + \delta D^{\alpha} g(t),$$

where γ and δ remain unchanged. In addition, the following essential features hold true for the Caputo fractional derivative:

$$\text{(i)} \quad D^{\alpha} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \quad 0 < \alpha < \beta+1, \beta > -1$$

$$\text{(ii)} \quad J^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} f^{k}(0^{+}) \frac{t^{k}}{k!}, \quad m-1 < \alpha \leq m, m \in \mathbb{N}$$

$$\text{(iii)} \quad D^{\alpha} C = 0, \quad \text{where} \quad C \text{ is a constant.}$$

Since there is no unique solution to a fractional differential equation without extra conditions being specified, we looked at these derivatives in the Caputo sense in this study. The extra conditions for the Caputo fractional differential equations are the standard ones, which are similar to the requirements for classical differential equations and are therefore known to us.

HAAR WAVELETS AND THEIR CONSTRUCTION

Haar wavelets are introduced as one sort of orthonormal wavelets after a short explanation of the multiresolution analysis that will be used in their creation.

Wavelet analysis is a strong and valuable mathematical technique with a very recent history. Harmonic analysis, signal and image processing, differential and integral equations, sampling theory, turbulence, geophysics, statistics, economics, finance, and medicine are just few of the many domains that have found use for the transform since it was codified into a rigorous mathematical framework. One definition of a wavelet is as a function with real numbers.ψ(t) It is suitable for the purposes:

$$\int_{-\infty}^{\infty} \psi(t)\,dt = 0, \quad \text{and} \quad \int_{-\infty}^{\infty} |\psi(t)|^2\,dt = 1.$$

If we accept the first premise, then ψ(t) must be a random variable, and the second requirement guarantees that the wavelet function has a value of 1 for its energy. Specifically, we may characterize wavelets as

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\,\psi\left(\frac{t-b}{a}\right), \quad a \neq 0, b \in \mathbb{R}, \qquad \textbf{(3)}$$

where a and b stand for the expansion and contraction factors, respectively. The signal's high-frequency components are represented by small values of a, while the low-frequency components are represented by big values of a. Furthermore, we have the following family of discrete wavelets where the parameters a and b are limited to discrete values as a = a j 0, b = kb0a j 0, a0 > 0, and b0 > 0.

$$\psi_{j,k}(t) = |a_0|^{j/2}\,\psi\left(a_0^j t - kb_0\right), \qquad \textbf{(4)}$$

where ψj,k establishing a wavelet foundation for L 2 (R). When a0 = 2 and b0 = 1, in particular, the functions ψj,k build upon an orthogonal framework. As in, ⟨ ψj,k ,ψm,n ⟩ = δ j,mδk,n.

Multiresolution analysis (MRA), first proposed by S. Mallat [151], is a powerful tool for building wavelet bases. The concept of using a generic formalism for building an orthogonal foundation of wavelets is quite astounding. In fact, MRA is essential in developing any kind of wavelet base. An MRA is a growing set of closed subspaces in mathematics. { Vj : j ∈ Z } of L 2 (R) conforming to the criteria (i) Vj ⊂Vj+1, j ∈ Z, (ii) ∪ j∈ZVj is dense in L 2 (R) and ∩ j∈ZVj = {0}, (iii) f(t) ∈ Vj if and only if f(2t) ∈ Vj+1, and (iv) there is a function ϕ ∈V0 called the scaling function, such that {ϕ(t −k) : k ∈ Z} form an orthonormal basis for V0. In view of the translation invariant property (iv), it is possible to generate a set of functions ϕ j,k in Vj , j ∈ Z, such that { ϕ j,k = 2 j/2ϕ(2 j t −k) : j,k ∈ Z } forms an orthonormal basis for Vj , j ∈ Z

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Let Wj , j ∈ Z be the complementary subspaces of Vj in Vj+1. These subspaces inherit the scaling property of { Vj : j ∈ Z } , namely f(t) ∈ Wj if and only if f(2t) ∈ Wj+1. By virtue of this property, one can find a function ψ ∈ W0 such that {ψ(x−k) : k ∈ Z} constitutes an orthonormal basis for L 2 (R), and hence, { ψj,k = 2 j/2ψ(2 j x−k) : j, k ∈ Z } will form an orthonormal basis for the subspaces Wj , j ∈ Z. Since, Wj's are dense in L 2 (R), therefore, it follows that the collection of functions { ψj,k : j, k ∈ Z } will form an orthonormal basis for L 2 (R). It is called an orthonormal wavelet basis with mother wavelet ψ. For the theoretical and mathematical treatment of wavelets, the reader is referred in 1910; A. Haar created the Haar wavelet function, which takes the form of a periodic pulse pair. As the earliest and most basic orthonormal wavelet, the Haar's compact support in [0,1] is its defining characteristic. The Haar scaling function, which looks like a square wave across the interval, is the simplest and most fundamental version of the Haar wavelet. t ∈ [0,1) as

$$h_0(t) = \begin{cases} 1, & \text{for } 0 \le t < 1, \\ 0, & \text{elsewhere.} \end{cases} \quad (5)$$

The preceding expression, known as the father wavelet, is the zeroth level wavelet that lacks both unit displacement and dilation. A mother wavelet, corresponding to the father wavelet, is defined as

$$h_1(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \\ 0, & \text{elsewhere.} \end{cases} \quad (6)$$

The interval of this mother wavelet may be halved by compressing it using a linear combination of the Haar scaling function and a translation.

$$h_1(t) = h_0(2t) - h_0(2t-1). \quad (7)$$

The other wavelet levels may also be created from h1(t) using the same translation and dilation methods. The family of Haar wavelets has a general formula that looks like this

$$h_i(t) = h_i(2^j t - k) = \begin{cases} 1, & \frac{k}{2^j} \le t < \frac{k+0.5}{2^j} \\ -1, & \frac{k+0.5}{2^j} \le t < \frac{k+1}{2^j} \\ 0, & \text{elsewhere,} \end{cases} \quad (8)$$

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where i = 1,2,...,m − 1,m = 2 The greatest resolution is denoted by the number M, where M is the largest positive integer that may be used. The integer decomposition of the index i is represented here by the symbols j and k. i = k + 2 j − 1, 0 ≤ j < i and 1 ≤ k < 2 j + 1. You may read more about Haar wavelets and their uses in a variety of fields in the aforementioned scholarly works.

CONVERGENCE OF THE HAAR WAVELET

Assume that y(t) satisfies a Lipschitz condition on [0,1), there exist positive number K > 0, such that |y(t1)−y(t2)| ≤ K|t1 − t2|, ∀t1,t2 ∈ [0,1), The Lipschitz constant K is used here. Then, we can write down ym(t) as an approximation of y(t) using the Haar method:

$$y_m(t) = \sum_{i=0}^{m-1} c_i h_i(t), \quad m = 2^{p+1},\ p = 0,1,2,\dots,M. \qquad \textbf{(9)}$$

Next, we may define the m-th-level mistake as

$$\|y(t) - y_m(t)\|_2 = \left\| y(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right\|_2 = \left\| \sum_{i=2^{p+1}}^{\infty} c_i h_i(t) \right\|_2 \qquad \textbf{(10)}$$

If we know the precise solution to a fractional-order differential equation, we may assess the inaccuracy. Our suggested method's convergence is described below.

Proposition 2.1 Assuming y(t) is Lipschitz-conformal on [0,1] and that ym(t) are Haar approximations of y(t), we get the following error bound:

$$\|y(t) - y_m(t)\|_2 \le \frac{K}{\sqrt{3}m^2}.$$

Proof. We have taken use of the Haar wavelets' inherent orthonormality to

$$\|y(t) - y_m(t)\|_2^2 = \int_0^1 |y(t) - y_m(t)|^2 dt = \sum_{r=2^{p+1}}^{\infty} \sum_{s=2^{p+1}}^{\infty} c_r \overline{c_s} \int_0^1 h_r(t)\overline{h_s(t)}\, dt = \frac{1}{m} \sum_{r=2^{p+1}}^{\infty} |c_r|^2.$$

The relation Eq.(may be used to get a rough approximation of the Haar wavelet coefficients cr's.

$$c_r = \int_0^1 y(t) h_r(t)\, dt = \frac{2^{j/2}}{\sqrt{m}} \left\{ \int_{I_1} y(t)dt - \int_{I_2} y(t)dt \right\}, \qquad \textbf{(11)}$$

where $I_1 = \left(\frac{k-1}{2^{-j}}, \frac{k-(1/2)}{2^{-j}} \right]$ and $I_2 = \left(\frac{k-(1/2)}{2^{-j}}, \frac{k}{2^{-j}} \right]$. Using mean value theorem of integrals, one can find $t_1 \in I_1$ and $t_2 \in I_2$ such that

$$c_r = \frac{2^{j/2}}{\sqrt{m}} \left\{ y(t_1)|I_1| - y(t_2)|I_2| \right\} = \frac{2^{-j/2-1}}{\sqrt{m}} \left[y(t_1) - y(t_2) \right].$$

Therefore,

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$$|c_r|^2 = \frac{2^{-j-2}}{m}|y(t_1) - y(t_2)|^2. \quad \textbf{(12)}$$

Using mean value theorem of derivative, there exist ξ , t1 ≤ ξ < t2 such that

$$|c_r|^2 = \frac{2^{-j-2}}{m}|t_1 - t_2|^2 y'^2(t)(\xi) \leq \frac{2^{-j-2}}{m}2^{-2j}K^2 = \frac{2^{-3j-2}K^2}{m}. \quad \textbf{(13)}$$

OPERATIONAL MATRIX OF THE GENERAL ORDER INTEGRATION

While Haar wavelets show promise in some contexts, they cannot be used directly to solve differential equations because of the discontinuity at their breaking points. F.C. Chen and C.H. Hsiao proposed a Walsh function-based operational matrix method to wavelet analysis as a means of resolving this shortcoming. The operational technique is distinguished by its primary feature, the reduction of a system of differential equations to a system of algebraic equations. Therefore, Chen and Hsiao provide a close approximation for the integration of the vector Hm(t) = [h0(t),h1(t),...,hm1(t)]T.

$$\int_0^t H_m(\tau)d\tau \cong QH_m(t), \quad \textbf{(14)}$$

In this case, Q is the m-order Haar wavelet operational matrix of integration. Using block pulse functions in a single framework, M. Yi and J. Huang provided a novel approach to determine the operational matrices of integration and differentiation for all orthogonal functions.

Without resorting to block pulse functions, we will calculate the general order integration Haar wavelet operational matrix using Definition 2.Fractional order integration using the Haar operational matrix The formula for Q is

$$\begin{aligned} Q^\alpha H_m(t) = J^\alpha H_m(t) &= \left[J^\alpha h_0(t), J^\alpha h_1(t), \ldots, J^\alpha h_{m-1}(t)\right]^T, \\ &= \left[Qh_0(t), Qh_1(t), \ldots, Qh_{m-1}(t)\right]^T, \end{aligned} \quad \textbf{(15)}$$

Where

$$Qh_0(t) = \frac{1}{\sqrt{m}}\frac{t^\alpha}{\Gamma(1+\alpha)}, \quad t \in [0,1], \quad \textbf{(16)}$$

CONCLUSION

Numerical solutions to one-dimensional singly perturbed boundary value problems are found using the Haar wavelet collocation technique. Several standard problems from the academic literature are used to evaluate the methodology. The numerical findings are compared to a selection of previously published techniques. Numerical data demonstrates the new method's advantage in terms of quick convergence and higher precision. The suggested approach may be used to swiftly and securely address a broad variety of related issues. Legendre wavelet and its features for addressing single starting or boundary value issues have been discussed in this study. The solution of algebraic equations is all that is required by using this technique. This approach has the potential to provide more precise numerical answers, and it also allows for control over the values of k and M. The provided examples demonstrate the validity and usefulness of this method, as well as the fact that just a modest value of k and M is required to get excellent outcomes. For both symbolic and numerical computations, mathematicians turn to MATLAB and MAPLE. In compared to other traditional approaches, it is clear that this strategy is very applicable, precise, and productive. In this study, we suggest a Chebyshev wavelet-based approach for solving Klein and Sine-Gordon equations that arise in many areas of science, engineering, and technology. The primary benefit of this approach is that the issue is reduced to an algebraic equation, making calculation quick and easy. There are three example cases used to evaluate the method's effectiveness and efficiency in the literature. The numerical findings are compared to a selection of previously published techniques. The numerical trials demonstrate the superiority of the spectral approach paired with Chebyshev wavelets.

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