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A STUDY OF NUMERICAL SCHEME FOR MATHEMATICAL FORMULATION ON INITIAL-BOUNDARY VALUE PROBLEMS

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ABSTRACT

In mathematics, in the field of partial differential equations, an initial value problem (also called the Cauchy problem) is a partial differential equation together with a specified value called the initial condition of the unknown function at a given point in the domain of the solution. In physics or other sciences, modelling a system frequently amounts to solving an initial value problem. Boundary value problems arise in several branches of physics, as any physical differential equation will have them. Problems involving the wave equation such as the determination of the normal nodes are often stated as boundary value problems. Boundary value problems are similar to initial value problems. A boundary value problem has conditions specified at the extremes (boundaries) of the independent variable in the equation, whereas an initial value problem has all the conditions specified at the same value of the independent variable (the value is at the lower boundary of the domain thus, the term initial value). To be useful in applications, an initial value problem as well as a boundary value problem should be well posed. This means that given the input to the problem, there exists a unique solution which depends continuously on the input.

KEYWORDS: Mathematical Formulation, Initial-Boundary, Value Problems, First Order PDES, Unbounded Domains, differential equations

INTRODUCTION

In this study, a finite difference scheme for first order hyperbolic equation in two independent variables x and t is developed. The first variable is typically space and the second variable usually represents time. A major challenge when designing finite difference scheme for hyperbolic equation is to design them to be stable while, at the same time ensuring that they do not damp out of the solution. In this proposed work, the numerical approximation schemes for initial and initial-boundary value problems are established.

Motivation

There is a vast literature on first order hyperbolic partial differential equations. Much effort has gone into devising the numerical approximation schemes for initial value problems and initial-boundary value problems. Gottlieb et al (1987), Bo (1998) and Coulombel (2009) studied the stability of finite difference schemes for first order hyperbolic initial-boundary value problems involving vector values functions in L2(IR+, IRN). In 1988, Warming and Beam studied the stability of semi- discrete approximations to the initial-boundary value problem

$$U_t = aU_x, \quad 0 \le x \le A, \ t \ge 0,$$
$$U(x,0) = u(x), \quad 0 \le x \le A,$$
$$U(A,t) = v(t), \quad t \ge 0, \tag{1}$$

where a > 0 with v(t) ≡ 0 in L2[0, A]. Sekino and Hamada (2008) developed the numerical solution of a advection equation ut + (a(x)u)x = 0 using wavelets. Despres (2009) and Teng (2010) established the finite difference schemes for the initial value problem

$$u_t + au_x = 0, \quad x \in I\!R, \ t \in I\!R^+,$$
$$u(x,0) = u_0(0), \quad x \in I\!R, \tag{2}$$

for discontinuous initial functions u0 which are bounded. These works derive the motivation to develop the numerical scheme for initial and initial-boundary value problem.

Problem Formulation

In this proposed work, the first model problem is an Initial Value Problem (IVP) on an infinite interval.

$$u_t = a(x)u_x(x), \quad x \in I\!R^+, \ t \in I\!R^+,$$
$$u(x,0) = u(x), \quad x \in I\!R^+, \tag{3}$$

where a(x) > 0 for all x ∈ IR+ and u = u(x) is some given function that is called as the initial condition and u ∈ C(IR+). Equation (3) is the model for wave propagation in homogeneous media.

The second model problem is called as Initial-Boundary Value Problem (IBVP) defined as

$$U_t = -aU_x, \quad x \in [0,1], \ t \in I\!R^+,$$
$$U(x,0) = u(x), \quad x \in [0,1],$$
$$U(0,t) = v(t), \quad t \in I\!R^+. \tag{4}$$

In this case, assume that a > 0 and that a boundary condition v(t) is given when x = 0. This is correct boundary condition because information is coming from left to right and u ∈ C[0, 1] and v ∈ C[0, ∞) satisfying the compatibility condition u(0) = v(0).

The main difference between the IVP (3) and IBVP (4) is the presence of a boundary condition. This latter condition is needed in many situations.

In this research work, to develop the fully discrete convergent numerical scheme for the problems IVP (3) and IBVP (4), semigroup theory was very much used. Semigroup theory provided an elegant method for constructing a solution to the initial-boundary value problems.

PRELIMINARIES

This section is devoted to some preliminary definitions and special case of Theorem of Pazy (1983) which was used as the major tool in this work.

Theorem 1. Let X be a Banach space with norm ‖ ‖. Assume that D(A) is dense in X, A : D(A) −→ X is a linear map and there is a λ with ℜ(λ) > 0 such that the range of λI − A is dense in X. Suppose that Xn are Banach spaces with norms ‖ ‖n. Further, for every n ≥ 1 there exist bounded linear operators Pn : X → Xn and En : Xn → X such that

(i) $||P_n|| \leq C_1$, $||E_n|| \leq C_2$, with C_1 and C_2 are constants independent of n.

(ii) $||P_n x||_n \to ||x||$ as $n \to \infty$ for every $x \in X$.

(iii) $||E_n P_n x - x|| \to 0$ as $n \to \infty$ for every $x \in X$.

(iv) $P_n E_n = I_n$, where I_n is the identity operator on X_n.

Let F (τn) be a sequence of bounded linear operators from Xn into Xn satisfying

$$||F(\tau_n)^k|| \leq 1. \tag{5}$$

Besides, the bounded linear maps

$$A_n = \rho_n^{-1}(F(\rho_n) - I)$$

have the property that

$$\mathbf{D}(A) = \{x \in X : E_n A_n P_n x \text{ converges}\}$$

and that

$$\lim_{n\to\infty} E_n A_n P_n x = Ax \tag{6}$$

for all x ∈ D(A). Then A¯, the closure of A is the infinitesimal generator of a contraction semigroup S(t) on X. Moreover, if knτn → t as n → ∞, then

$$\lim_{n\to\infty} ||F(\tau_n)^{k_n} P_n x - P_n S(t)x||_n = 0.$$

In the sequel, the term solution refers to a generalized solution in an appropriate sense.

If $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k)$, then let the notation $\alpha(i) = \alpha_i$.

For $x \in I\!R$, $\lfloor x \rfloor = \sup\{n \in \mathbf{Z} : n \leq x\}$.

Theorem 2 (Hille-Yosida Theorem). A linear (unbounded) operator A is the infinitesimal generator of a C0 semigroup of contractions T (t), t ≥ 0 if and only if,

(i) A is closed and $\overline{D(A)} = X$.

(ii) The resolvent set ρ(A) of A contains IR+ and for every λ > 0,

$$||R(\lambda : A)|| \leq \frac{1}{\lambda}. \tag{7}$$

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EXACT SOLUTION FOR INITIAL VALUE PROBLEM AND INITIAL-BOUNDARY VALUE PROBLEM

This section is concerned with the exact solution for initial and initial-boundary value problem considered in this work.

Exact Solution for the Initial Value Problem

It is well known that the solution to (3) is given by

$$u(x,t) = u(\beta^{-1}(t + \beta(x)),$$

where $\beta(x) = \int_0^x \frac{d\xi}{a(\xi)}$.

Here, the aim was to compute the solution u(x, t) of (3) which was not necessarily bounded, numerically on any bounded domain making use of the values of u(x, t) on a bounded domain. The following theorem facilitates this result.

Theorem 1. Assume that $a \in C[0, \infty)$ and $a(x) > 0$ for all $x \in IR+$. Let $M > 0$ and $T > 0$. Define $a_M : [0, M] \longrightarrow IR+$ as

$$\begin{aligned} a_M(x) &= a(x), \quad 0 \le x \le M - \frac{1}{M}, \\ &= a\left(M - \frac{1}{M}\right)\sqrt{M(M-x)}, \quad M - \frac{1}{M} \le x \le M \end{aligned}$$

and let $f \in C[0, M]$. The solution to the problem

$$\begin{aligned} \frac{\partial V}{\partial t} &= a_M(x)\frac{\partial V}{\partial x}, \quad 0 \le t \le T, \quad 0 \le x \le M, \\ V(x,0) &= f(x), \quad 0 \le x \le M, \\ V(M,t) &= f(M) \end{aligned} \tag{8}$$

exists, unique and is given by

$$V(x,t) = f(\beta_M^{-1}\left[Min\left(t + \beta_M(x), \beta_M(M)\right)\right], \quad \text{where}$$

$$\begin{aligned} \beta_M(x) &= \int_0^x \frac{d\xi}{\alpha(\xi)}, \quad 0 \le x \le M - \frac{1}{M}, \\ &= \int_0^{M-\frac{1}{M}} \frac{d\xi}{\alpha(\xi)} + \int_{M-\frac{1}{M}}^x \frac{d\xi}{\alpha(M-\frac{1}{M})\sqrt{M(M-x)}}, \quad M - \frac{1}{M} \le x \le M. \end{aligned}$$

Further,

$$S_t f(x) = f(\beta_M^{-1}\left[Min\left(t + \beta_M(x), \beta_M(M)\right)\right])$$

defines a contraction semigroup on C[0, M] whose generator is given by

$$\mathbf{D}(A) = \left\{ g \in \mathbf{C}[0, M] : g' \in \mathbf{C}[0, M) \quad \text{and} \quad \lim_{x \to M} a_M(x) g'(x) = 0 \right\}$$

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and

$$\begin{aligned} Ag(x) &= a_M(x)g'(x), \quad x \in [0, M), \\ Ag(M) &= 0. \end{aligned}$$

Further, choosing M > N such that

$$\sup_{t\in[0,T],x\in[0,N]} (t + \beta(x)) < b\left(M - \frac{1}{M}\right),$$

$$V(x,t) = u(x,t), \quad (x,t) \in [0,N] \times [0,T] \tag{9}$$

provided f ∈ C[0, M] is the restriction of u to [0, M].

Proof. Define for $t \geq 0$, $T_t : [0, M] \longrightarrow [0, M]$ as

$$T_t x = \beta_M^{-1}\left[Min\left(t + \beta_M(x), \beta_M(M)\right)\right].$$

Now, it is easy to show that $T_{s+t} = T_s \circ T_t$.

$$\begin{aligned} T_s \circ T_t x &= \beta_M^{-1}\left[Min\left(s + \beta_M(T_t x), \beta_M(M)\right)\right] \\ &= \beta_M^{-1}\left[Min\left(s + \beta_M(\beta_M^{-1}\left[Min\left(t + \beta_M(x), \beta_M(M)\right)\right]), \beta_M(M)\right)\right] \\ &= \beta_M^{-1}\left[Min\left(s + \left[Min\left(t + \beta_M(x), \beta_M(M)\right)\right], \beta_M(M)\right)\right] \\ &= \beta_M^{-1}\left[Min\left(\left[Min\left(s + t + \beta_M(x), s + \beta_M(M)\right)\right], \beta_M(M)\right)\right] \\ &= \beta_M^{-1}\left[Min\left(s + t + \beta_M(x), \beta_M(M)\right)\right] \\ &= T_{s+t}x. \end{aligned}$$

Also, it is easy to say that St is a semigroup, since $S_t f(x) = f(T_t x)$.

It is obvious that kStfk ≤ kfk and hence St is a contraction semigroup.

Now, by Hille-Yosida Theorem, if B is the generator of S_t then

$$\begin{aligned} (I - B)^{-1}h(x) &= \int_0^\infty e^{-t} S_t h(x)dt \\ &= \int_0^\infty e^{-t} h(\beta_M^{-1}\left[Min\left(t + \beta_M(x), \beta_M(M)\right)\right])dt \\ &= \int_0^{\beta_M(M)-\beta_M(x)} e^{-t} h(\beta_M^{-1}(t + \beta_M(x)))dt + \int_{\beta_M(M)-\beta_M(x)}^\infty e^{-t} h(N)dt \\ &= \int_x^M e^{\beta_M(x)-\beta_M(y)} \frac{h(y)}{a_M(y)}dy + h(N)e^{\beta_M(x)-\beta_M(M)}, \end{aligned}$$

where $y = \beta_M^{-1}(t + \beta_M(x))$.

Now, consider the differential equation

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$$f(x) - a_M(x)f'(x) = h(x), x \in [0, M),$$
$$f(M) = h(M),$$

which is equivalent to

$$f(x) - a(x)f'(x) = h(x), x \in [0, M),$$
$$\lim_{x\to M} a(x)f'(x) = 0.$$

Since $\int_0^M \frac{ds}{a_M(s)} < \infty$ for every h ∈ X, there is a unique solution f ∈ D(A) to the above differential equation which is given by

$$f(x) = e^{\int_0^x \frac{ds}{a_M(s)}} \int_x^M \frac{h(y)}{a_M(y)} e^{-\int_0^y \frac{ds}{a_M(s)}} dy$$
$$+h(M)e^{-\int_0^M \frac{ds}{a_M(s)}} + \int_0^x \frac{ds}{a_M(s)}$$
$$= \int_x^M e^{\beta_M(x)-\beta_M(y)} \frac{h(y)}{a_M(y)} dy + h(N)e^{\beta_M(x)-\beta_M(M)}.$$

Hence it can be shown for the operators A and B, (I − A) −1 = (I − B) −1 . From this, one can easily conclude that D(A) = D(B) and for all g ∈ D(A), Bg = Ag.
As β is a strictly increasing function by (3.9), for t ∈ [0, T] and x ∈ [0, N] then

$$x \le \beta^{-1}(t + \beta(x)) < M - \frac{1}{M}.$$

Hence

$$\beta(\beta^{-1}(t + \beta(x)) = \beta_M(\beta_M^{-1}(t + \beta_M(x))).$$

From this, it is concluded that Stf(x) = V (x, t) = u(x, t) for all x ∈ [0, N] and t ∈ [0, T].

Exact Solution for the Initial-Boundary Value Problem

Theorem 2. Let u ∈ C[0, 1] and v ∈ C[0, ∞) be such that u(0) = v(0). Define u0(x) = u0(x) − u0(0). If U is a solution to (4), U is a solution to

$$\overline{U}_t = -a\overline{U}_x, \quad 0 \le x \le 1, t \ge 0,$$
$$\overline{U}(x, 0) = u_0(x), \quad 0 \le x \le 1, \qquad (10)$$

and V is a solution to

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$$V_x = \frac{-1}{a} V_t, \quad 0 \le x \le 1, t \ge 0,$$
$$V(x,0) = u_0(0), \quad 0 \le x \le 1,$$
$$V(0,t) = v(t), \quad t \ge 0,$$
(11)

then U(x, t) = U(x, t) + V (x, t). Further, fixing T > 0, define the contraction semigroups St : X → X where X = {u ∈ C[0, 1] : u(0) = 0} and Tx : Y −→ Y
where Y = C[0, T] as

$$(S_t u_0)(x) = u_0(x - at), \quad at \le x \le 1,$$
$$= 0, \quad 0 \le x \le at,$$

$$T_x w(t) = w(0), \quad 0 \le t \le x/a,$$
$$= w\left(t - \frac{x}{a}\right), \quad t \ge x/a.$$

Then U(x, t) = Stu0(x) + Σxw(t) for all (x, t) ∈ [0, 1] × [0, T], where w is the restriction of v to [0, T].
Further, if A and B denote the generators of St and Tx respectively, then
D(A) = {g ∈ X : g ′ ∈ X} , D(B) = {g ∈ Y : g ′ ∈ Y and g ′ (0) = 0} , Ag = −ag′ for all g ∈ D(A) and Bg = −1 a g ′ for all g ∈ D(B).

CONVERGENT NUMERICAL SCHEME FOR THE INITIAL VALUE PROBLEM AND INITIAL-BOUNDARY VALUE PROBLEM

This section explains the convergent numerical scheme for the initial and initial-boundary value problems. For the initial value problem, one can obtain a modified problem posed on a bounded domain whose solution exactly coincides with the solution of the original problem on a smaller bounded domain. The numerical solution to the modified problem converges to the solution of the original problem on the smaller bounded domain. For the initial-boundary value problem, one can present the discrete semigroup approximations after decomposing it into two problems each of which gives rise to a semigroup.

A Convergent Numerical Scheme for the IVP

For the initial value problem (3), for every subset [0, N] × [0, T] ⊂ IR+ × IR+, one can obtain M > N and an initial value problem posed on [0, M] × [0, T] whose solution exactly coincides with the solution of (3) on [0, N] × [0, T]. Then one develops a finite difference scheme converging to the solution of the problem posed solution to (3) on [0, M] × [0, T] which converges to the solution to (3) on [0, N] × [0, T].
The following theorem facilitates this result.

Theorem 1. Let X = C[0, M] and A be as in Theorem 1. Let Xn = IRn+1 whose elements are denoted as α = (α0, α1, . . . , αn). Both spaces X and Xn are normed with the usual supremum norm. Now define

Pn : X → Xn as (Pnf)i = f (iM/n), i = 0, 1, . . . , n.

En : Xn → X as

En(α) is the piecewise linear function with En(α)(iM/n) = αi. Let

$$\tau_n = \frac{1}{2n \sup\limits_{x\in[0,M]} |a(x)|}.$$

Define an operator F (τn) : Xn → Xn as

$$\begin{aligned}(F(\tau_n)\alpha)_i &= (1 - n\tau_n a_M(iM/n))\,\alpha_i + n\tau_n a_M(iM/n)\alpha_{i+1}, \quad i = 0, 1, \ldots, n-1\\ &= \alpha_n, \quad i = n.\end{aligned}$$

Choosing kn = $\lfloor \frac{t}{\tau_n} \rfloor$, it can be shown that

$$||F(\tau_n)^{k_n} P_n f - P_n S(t) f||_n \to 0 \quad \text{as} \quad n \to \infty. \tag{12}$$

In particular, fixing t ∈ [0, T] and x ∈ [0, N],

$$\lim_{n\to\infty} F(\tau_n)^{k_n} P_n f(\lfloor nx/M \rfloor) = u(x, t), \tag{13}$$

where u(x, t) is the solution to (3).

Proof. Pn is obviously linear and ||Pn|| ≤ 1. From the definitions of the norms and the uniform continuity of the elements of X, it is also clear that (ii) of Theorem 1 is satisfied. One can easily obtain that ||En|| ≤ 1.

Obviously, PnEn = In and (iii) of Theorem 1 follows from the uniform continuity of the element of X and the definitions of En and Pn. Now, it is easy to formulate the difference equation giving rise to the definition of F (τn). Consider for each given n and τn functions defined on the lattice (iM/n, jτn), i = 0, 1, 2, . . . , j = 0, 1, 2, . . . in the (x, t) plane.

Let V (iM/n, jτn) = ui,j . Noting that sup x∈[0,M] aM(x) > 0, τn is well defined. Consider the difference equation corresponding to the differential equation in (8) is

$$\begin{aligned}\frac{u_{i,j+1} - u_{i,j}}{\tau_n} &= a_M(iM/n)n(u_{i+1,j} - u_{i,j}), \quad i = 0, 1, 2, \ldots, n-1,\\ \frac{u_{n,j+1} - u_{n,j}}{\tau_n} &= 0, \quad \text{which can be simplified as}\end{aligned}$$

$$\begin{aligned}u_{i,j+1} &= (1 - n\tau_n a_M(iM/n))\,u_{i,j} + n\tau_n a_M(iM/n)u_{i+1,j}, \quad i = 0, 1, \ldots, n-1,\\ u_{n,j+1} &= u_{n,j}.\end{aligned}$$

If ui,0 = fi are given, then one can compute all ui,j by the above recursion formula. Let fi = f(iM/n).

Now,

$$
\begin{aligned}
&\|F(\tau_n)(\alpha)\|_n \\
&\quad = \max\left(\max_{0\le i\le n-1} |(1 - n\tau_n a_M(iM/n))\,\alpha_i + n\tau_n a_M(iM/n)\alpha_{i+1}|\,, |\alpha_n|\right) \\
&\quad = \max\left(\max_{0\le i\le n-1} (1 - n\tau_n a_M(iM/n) + n\tau_n a_M(iM/n))\max(|\alpha_i|, |\alpha_{i+1}|), |\alpha_n|\right) \\
&\quad = \max(|\alpha_0|, |\alpha_1|, \ldots, |\alpha_n|) \\
&\quad = \|\alpha\|_n.
\end{aligned}
$$

Therefore, ||F(τn)|| ≤ 1 and the stability condition (5) of Theorem 1 holds.

For $f \in D$,

$$
\begin{aligned}
&\|\tau_n^{-1}(F(\tau_n) - I)P_n f - P_n a_M(iM/n)f'\|_n \\
&\quad = \sup_i \left|\frac{a_M(iM/n)}{n}\left(f((i+1)M/n) - f(iM/n)\right) - a_M(iM/n)f'(iM/n)\right|.
\end{aligned}
\tag{14}
$$

Since f ∈ D, Af is uniformly continuous on [0, M] and therefore, the right hand side of (14) tends to zero as n → ∞. Hence (3.6) of Theorem 1 is satisfied.

Finally, Theorem 1 is applied to this problem, one has to show that for some λ > 0 the range λI − A is dense in X. But in Theorem 1 it is already shown that the range of λI − A is the whole of X. Also, using the expression of (I − B) −1 in of Theorem 1, ||(I − A) −1h|| ≤ khk.

CONCLUSION

In this short final study, we give an overall assessment of the logic of partial functions, as currently understood, and make some suggestions for future research. First, we reiterate what was said in the introduction to this thesis: that partial functions have, in general, more favourable logical and computational properties than binary relations. The results in this thesis only reinforce this viewpoint. Consider those operations with a first-order definition—by which we mean definable in the manner required by the fundamental theorem, It had already been established that when considering these types of operations, generally the representation classes are finitely axiomatisable and have equational theories of low complexity, the finite representation property is satisfied, and representability of finite algebras is simple to decide. And it had been found that these remarks extend to multiplace functions as well. This is all in contrast to how relations behave. If we are to reason about programs specified by code written in any general-purpose (that is to say, Turing-complete) language, then we are certainly going to need to be able to express some kind of un- bounded iteration operation. However, obtaining results by translating in this way necessarily requires antidomain in the signature. If we have in mind to model partial recursive functions without any restrictions, then it is difficult to justify including antidomain, as identifying the points where partial recursive functions are undefined is not in general and effectively computable operation and so not expressible in any programming language.

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