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Description of the set of strictly regular quadratic bistochastic operators and examples

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Abstract: The present paper focuses on the dynamical systems of the quadratic bistochastic operators (QBO's) on the standard simplex. We show the character of connection of the dynamical systems of a bistochastic operator with the dynamical systems of the extreme bistochastic operators. Moreover, we prove that almost all quadratic bistochastic operators is strictly regular and give description of the strictly regular quadratic bistochastic operators in the convex polytope of QBO's. Furthermore, convexity of the set of strictly regular QBO's and its density in the set of QBO's is proven and nontrivial examples to strictly regular bistochastic operators are given.

Keywords: Affine hull, convex hull, simplex, extreme point, relative interior of a convex set, fixed point, periodic point, stochastic operator, bistochastic operator, strictly regular stochastic operator.

Introduction

A lot of genetic processes in population genetics can be associated with some nonlinear dynamical systems. Dynamical systems which are generated by quadratic stochastic operators (QSOs) appear many problems of mathematical genetics. Generally, dynamical systems of QSOs are very complex and difficult. Therefore, dynamical systems of certain type QSOs are investigated. Quadratic bistochastic operators are one of type of QSOs. An interesting property of dynamical systems of QBOs is that trajectory of any initial point converges some periodic orbit. In other words, ω−limit set of any initial point is always finite.

The present paper is appeared in the intersection of several branches of mathematics like the theory of convex polytopes, majorization theory and theory of QSO's. In the paper, we give algebraic expression of the relative interior points of convex polytopes and prove a theorem about periodic points of bistochastic operators, furthermore, we prove strictly regularity of all operators in the relative interior of QBOs' poytope and give non trivial examples to strictly regular quadratic bistochastic operators.

2. Preliminaries

In this section we provide some important definitions in the theory of convex polytopes, majorization theory, and theory of QSOs in order to give theorems in next sections. Therefore, we recall some concepts in affine geometry and theory of dynamical systems in this section. Initially, we define affine structure on R^d. In the meantime, we do not differ the concept of point from vector in R^d and this does not bring confusions.

The combination $\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_s a_s$ is called affine (convex) combination of $a_1, a_2, \ldots, a_s \in R^d$ when $\sum_{j=1}^{s} \lambda_j = 1$ where $\lambda_j \in R$ ($\lambda_j \in R_+$) for $j = \overline{1,s}$. Nonempty subset $L \subset R^d$ is called affine subspace of R^d if it is closed w.r.t. affine combinations of its elements. Clearly, nonempty intersection of affine spaces is also an affine space, whence affine hull, $Aff(M)$ of a subset M of R^d is defined as the intersection of all affine subspaces which includes M. It can be easily proved that

$$Aff(M) = \left\{ \sum_{i=1}^{t} \mu_i z_i : \mu_i \in R, t \in N, \sum_{i=1}^{t} \mu_i = 1, z_i \in M, i = \overline{1;t} \right\} \quad (2.0.1)$$

The points $a_1, a_2, \ldots, a_r \in R^d$ is called affine dependent if one of them lies in the affine hull of the others. Otherwise, $a_1, a_2, \ldots, a_r$ is

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called affine independent. Maximal affine independent system of elements of affine space L is called affine basis of L. Evidently, number of elements in the basis does not depend on its choosing. Next small cardinality from the cardinality of the affine basis is called affine dimension of L and denoted by $\dim(L)$. A subset $Q \subset R^d$ is called convex if it is closed w.r.t. convex combinations of its elements and empty set is considered as convex set by the definition. Intersection of convex sets is also convex, whence convex hull, $conv(M)$, of a given nonempty subset

$M \subset R^d$ is defined as the intersection of all convex sets which includes it. Obviously,

$$conv(M)=\left\{\sum_{i=1}^{t}\mu_i z_i : \mu_i \in R_+, t \in N, \sum_{i=1}^{t}\mu_i = 1, z_i \in M, i=\overline{1;t}\right\}.$$

. A point $v \in Q$ is called extreme point of Q if $\lambda x+(1-\lambda)y=v$, $\lambda \in (0;1)$, $x,y \in Q$ implies $x=y=v$ and the set of all extreme points of Q is denoted by $Extr(Q)$. Convex hull of finite set is called polytope. Simplex is defined as the convex hull of affine independent vectors of R^d. The following set is called standard $(d-1)-$ dimensional simplex;

$$S^{d-1}=\left\{x=(x_1,x_2,\dots,x_d)\in R^d : \sum_{i=1}^{d}x_i = 1, x_i \geq 0, i=\overline{1;d}\right\}.$$

In the paper we consider l_1 norm in R^d, namely $\|x\|=|x_1|+|x_2|\dots+|x_d|$ for $x=(x_1,x_2,\dots,x_d)\in R^d$. Then a metrics in $Aff(M)$ can be induced from this norm where $M \subset R^d$ nonempty subset. The interior of M w.r.t. this induced metric is called relative interior of M and it is denoted by $ri(M)$. We mention that relative interior of a set does not depend of choosing norm in R^d because of all norms in R^d are mutually equivalent, so theygenerate the same topology.

For any $x=(x_1, x_2, \dots, x_m)\in S^{m-1}$ due to **[1]**, we define $x_{\downarrow}=(x_{[1]}, x_{[2]}, \dots, x_{[m]})$, here $x_{[1]}\geq x_{[2]}\geq \dots \geq x_{[m]}$ - non-increasing rearrangement of coordinates of x. The point $x_{\downarrow}$ is called rearrangement of x by non-increasing. For two elements x, y taken from the simplex S^{m-1}, we say that the element x majorized by y (y majorates x), and write $x \prec y$ (or $y \succ x$) if the following hold:

$$\sum_{i=1}^{k}x_{[i]} \leq \sum_{i=1}^{k}y_{[i]}$$

for any $k=\overline{1;(m-1)}$.

Geometric illustration can be given to majorization as follows: we call permutation vector of y such vector that generate from permutating places of coordinates of y and let Π_y be the convex hull of all permutation vectors of y. Then due to [1] the following hold

PROPOSITION 2.1. [1]. *$x \prec y$ if and only if $x \in \Pi_y$. Furthermore, all permutation vectors of y are extreme points of Π_y.*

A continuous operator $V: S^{m-1}\to S^{m-1}$ is called $m-$dimensional stochastic operator. We call an operator $V: S^{m-1}\to S^{m-1}$ quadratic stochastic operator (QSO) if it has the following form:

$$V(x)_k=\sum_{i,j=1}^{m}p_{ij,k}x_i x_j \text{, for } k=\overline{1;m}$$

where $x=(x_1, x_2, \dots, x_m)\in S^{m-1}$,

$p_{ij,k}=p_{ji,k}\geq 0$, $\sum_{r=1}^{m}p_{ij,r}=1$ for $\forall i, j, k \in \{1,2, \dots, m\}=N_m$. A quadratic stochastic operator is called evolution operator in population genetics and the coefficient $p_{ij,k}$ is called heredity coefficients of this operator.

Clearly, any QSO is a stochastic operator. By the form of QSO we deduce that any QSO associated with unique cubic stochastic matrix of certain type $\{p_{ij,k}\}$ in the space of real cubic matrices $M_n^c(R)$. Whence, according to this correspondence we can define (convex) addition between QSOs.

DEFINITION 2.1. A stochastic operator V is called *bistochastic*if $V(x) \prec x$, for $\forall x \in S^{m-1}$. A bistochastic QSO is called*quadratic bistochastic operator* (QBO). The set of all $m-$ dimensional QBOs is denoted by B_m.

According to the definition of majorization, B_m is closed set and it is also closed w.r.t. convex sum of its elements, therefore, it is closed, convex subset of $M_n^c(R)$. Extreme points of B_m is called extreme QBO and some necessary conditions and some sufficient conditions for extremity of a QBO was found in the doctoral thesis of R.Ganikhodjaev **[2]**, but any criterions did not find so far.

By the definition of majorization we have $V\left(\left(\frac{1}{m},\frac{1}{m},\dots,\frac{1}{m}\right)\right)=\left(\frac{1}{m},\frac{1}{m},\dots,\frac{1}{m}\right)$ for a QBO V, in other words barycenter of the simplex is a fixed point of any bistochastic operator. The following theorem characterizes main properties of bistochastic operators and B_m.

THEOREM 2.1. [**2**] *Let* $V: S^{m-1} \to S^{m-1}$ *be a quadratic bistochastic operator then:*

i) $|\omega_V(x_0)| < \infty$, *for* $\forall x_0 \in S^{m-1}$, *where* $\omega_V(x_0)$ *(*$\omega-$*limit set of* x_0*) is the set of limit points of* $\{V^n(x_0)\}_{n=0}^{\infty}$;

ii) $P \circ V$ *is quadratic bistochastic operator for any coordinate permutation operator* $P: S^{m-1} \to S^{m-1}$.

iii) $Extr(\mathrm{B}_m) < \infty$ *for* $m \in N$;

REMARK 2.1. Coordinate permutation operator is such operator that it maps a vector to its permutation vector which permutation order of places of coordinates does not change when vector is changing.

In the view of theory of dynamical systems, the dynamical system of a certain operator may have been very strange behavior. More simple dynamical system among such strange dynamical systems is that every trajectory in the dynamical system converge a point. In the theory of QSO's, operators which have such simple dynamical system are said regular.

DEFINITION 2.2. A QSO is called*regular* if its trajectories always converge. A regular QSO is called*strictly regular* if it has unique fixed point.

Hence the dynamical system of strictly regular QSO is simpler than dynamical system of regular ones. Some properties of regular QSO's are studied in ([5]-[8]). In particular, the following simple criterion for regularity of a bistochastic operator is given in [7] and [8].

THEOREM 2.2. ([7], [**8**]) *Let* $V: S^{m-1} \to S^{m-1}$ *be a bistochastic operator, then* V *is regular if and only if it does not have any order periodic points except fixed points.*

Obviously, the unique fixed point of a strictly regular bistochastic operator which is said in the definition is the barycenter of the simplex. Hence by the Theorem 2.2 we have quickly the following simple criterion for strictly regularity of the bistochastic operators.

PROPOSITION 2.2. *Quadratic bistochastic operator is strictly regular iff it* *have not any order periodic points except its unique fixed point* $\left(\frac{1}{m},\frac{1}{m},\dots,\frac{1}{m}\right)$.

3. Main results

3.1. On the relative interior of convex polytopes. In this subsection we recall some affine properties of convex sets and give algebraic expression of the points in the relative interior of the convex polytopes.

THEOREM 3.1. *Let* $f_1, f_2, \dots, f_s$ *be a points of* R^d *such that none does not lie convex hull of the others and* $Q = conv\{f_1, f_2, \dots, f_s\}$. *Then*

$$ri(Q) = \{\lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_s f_s : \sum_{j=1}^{s} \lambda_j = 1, \lambda_j > 0, j = 1; s\}$$

.

We use several lemmas in proving the theorem. For the sake of brevity we also use the following notations: let $A \subset R^d$, $x, y \in R^d$ and $\lambda \in R$ then $A + x := \{a + x : a \in A\}$, $\lambda A := \{\lambda a : a \in A\}$, $[x; y] := \{\mu x + (1-\mu) y : \mu \in [0; 1]\}$. Similarly, half open $[x; y)$, $(x; y]$ and open $(x; y)$ intervals is defined like $[x; y]$.

LEMMA 3.1. *Let* Q *be a nonempty convex subset of* R^d, *then* $ri(Q) \neq \varnothing$.

PROOF. We consider two cases in order to prove the lemma.

Special case: In this case we prove the lemma for the simplexes. Let $S \subset R^d$ be a d_0 − dimensional simplex (clearly $d_0 \le d$). Then according to the definition of simplex, there are affine independent vectors $v_1, v_2, \dots, v_{d_0+1} \in R^d$ such that $S = conv\{v_1, v_2, \dots, v_{d_0+1}\}$. Hence $v_1, v_2, \dots, v_{d_0+1}$ is an affine basis for $Aff(S)$ according to (2.0.1), so any element of $x \in Aff(S)$ can be uniquely expressed as $x = \mu_1 v_1 + \mu_2 v_2 + \dots + \mu_{d_0+1} v_{d_0+1}$ with $\mu_1 + \mu_2 + \dots + \mu_{d_0} = 1$. Therefore, the mapping $\varphi : Aff(S) \to Aff(S^{d_0})$ which is determined as

$$\varphi(\mu_1 v_1 + \mu_2 v_2 + \dots + \mu_{d_0+1} v_{d_0+1}) := (\mu_1, \mu_2, \dots, \mu_{d_0+1})$$

is well-defined. φ is a bijection and a continuous mapping between S and standard d_0−dimensional simplex S^{d_0}. Obviously, $G := \{(\mu_1, \mu_2, \dots, \mu_{d_0}) : \mu_j > 0\}$ is open in R^{d_0}, hence $G_1 = G \cap Aff(S^{d_0})$ is open in $Aff(S^{d_0})$. Since φ is bijection and continuous we have that

$$\varphi^{-1}(G_1) = \{\sum_{j=1}^{d_0} \mu_j v_j : (\mu_1, \mu_2, \dots, \mu_{d_0}) \in G_1, \sum_{j=1}^{d_0} \mu_j = 1\}$$

is open set in $Aff(S)$. Now, note that $\varphi^{-1}(G_1) \subset S$, moreover $\varphi^{-1}(G_1)$ is open. Therefore, $\varphi^{-1}(G_1) \subset ri(S)$ by the definition of relative interior, thus $ri(S) \neq \varnothing$.

General case: Let Q be a nonempty convex subset and $d_0 = dim(Aff(Q))$. In $d_0 = 0$ there is nothing to prove. So, we can assume $d_0 > 0$. Then there is $e_1, e_2, \dots, e_{d_0+1} \in Q$ which $\{e_1, e_2, \dots, e_{d_0+1}\}$ is a affine basis for $Aff(Q)$ by the expression (2.0.1). Let us consider the simplex $S := conv\{e_1, e_2, \dots, e_{d_0+1}\}$. Then $S \subset Q$ by the convexity of Q. So we have $Aff(S) = Aff(Q)$ by the $Aff(S) \subset Aff(Q)$ and $dim(Aff(S)) = d_0 = dim(Aff(Q))$. According to the proved statement in the special case we have $ri(S) \neq \varnothing$. Hence $\exists x_0 \in S$ and open neighborhood O_{x_0} of in R^d,

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such that $O_{x_0} \cap Aff(Q) \subset Q$. Whence according to $Aff(S) = Aff(Q)$ and $S \subset Q$ we have $O_{x_0} \cap Aff(Q) \subset Q$. The last inclusion implies that x_0 is a relative interior point of Q. □

LEMMA 3.2. *Let Q be a nonempty closed convex subset of R^d. Then for $x \in ri(Q)$ and $y \in Q / \{x\}$ the relation $[x; y) \subset ri(Q)$ holds.*

PROOF. Let $x_\lambda = \lambda x + (1-\lambda)y$ be a point in $(x; y)$. $x \in ri(Q)$ implies existence of such open neighborhood O_x of x that $O_x := O_x \cap Aff(Q) \subset Q$. . In R^d, the continuity of addition and multiplying to scalar is followed that $\lambda O_x + (1-\lambda)y$ is open set. Therefore, $(\lambda O_x + (1-\lambda)y) \cap Aff(Q) = \lambda O_x + (1-\lambda)y$ is open set of $Aff(Q)$ and convexity of Q implies $\lambda O_x + (1-\lambda)y \subset Q$. Whence $x_\lambda \in \lambda O_x + (1-\lambda)y \subset Q$ implies $x_\lambda \in ri(Q)$. □

COROLLARY 3.1. *For any nonempty closed convex subset Q of R^d we have* $Q = ri(Q)$.

PROOF. We get $\forall y \in Q$, then $ri(Q) \neq \varnothing$ by the Lemma 3.1. Hence we can get $\exists x \in ri(Q)$, thus by the Lemma 3.2 we have $[x; y) \subset ri(Q)$ (*). We consider an open ball O_y centered at y. Then $O_y \cap [x; y) \neq \varnothing$ and it is subset of $ri(Q)$ by the (*) relation. Hence, $y \in ri(Q)$. □

LEMMA 3.3. *The following two conditions are mutually equivalent for any convex closed set Q:*

i) $x \in ri(Q)$;

ii) For $\forall y \in Q / \{x\}$, there is $z \in Q$ such that $x \in (y; z)$.

PROOF. $i) \Rightarrow ii)$: Let $x \in ri(Q)$, then there is $\exists B_\delta(x)$ open ball with centered at x which

$$B_\delta(x) \cap Aff(Q) \subset Q \quad (3.1.1)$$

We get $\forall y \in Q / \{x\}$. Since $y \neq x$, we have $\| y - x \| \neq 0$. We consider the vector

$$z = -\frac{\delta}{2\| y-x \|} y + \left(1 + \frac{\delta}{2\| y-x \|}\right) x$$

and show that z is desired vector. Indeed, z belongs to $Aff(Q)$ as an affine combination of x, y. In the other hand, $\| z - x \| = \frac{\delta}{2} < \delta$, so $z \in B_\delta(x)$. Then by the (3.1.1) we have $z \in Q$. But the determination of z implies that

$$x = \frac{2\| y-x \|}{\delta + 2\| y-x \|} z + \frac{\delta}{\delta + 2\| y-x \|} y \in (y; z).$$

$ii) \Rightarrow i)$:We assume that $x \in Q$ is a point which satisfies the condition of the second statement. Since $ri(Q) \neq \varnothing$ by the Lemma 3.1 we can choose a point y in $ri(Q)$. Then there exist $\exists z \in Q$ that $x \in (y; z)$. According to Lemma3.2 $(y; z) \subset ri(Q)$. Hence we have $x \in ri(Q)$. □

With the above three lemmas at hand we can now pass to proving Theorem3.1.

PROOF. (Theorem 3.1) First we show that for any point x in $ri(Q)$ can be represented as a convex combinations of $f_1, f_2, \ldots, f_s$ which the convex representation includes each of f_j with positive coefficient. Since Lemma 3.3 we have $f_j \notin ri(Q)$ for $j = \overline{1;s}$. Therefore, $x \notin ExtrQ$. After that we consider

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sequentially the extreme points $f_1, f_2, \ldots, f_s$. Then Lemma 3.3 implies existence of such distinct points $z_1, z_2, \ldots, z_s$ in Q that $x \in (f_j; z_j)$ for $j = \overline{1;s}$. Algebraically, the last relations mean $\exists \mu_1, \mu_2, \ldots, \mu_s \in (0;1)$ which

$$x = \mu_j f_j + (1-\mu_j) z_j \quad (3.1.2)$$

for $j = \overline{1;s}$. Then after averaging these $s-$ equations in (3.1.2) we have

$$x = \frac{1}{s}\sum_{j=1}^{s}(\mu_j f_j + (1-\mu_j)z_j) = \sum_{j=1}^{s}\frac{\mu_j}{s}f_j + \sum_{j=1}^{s}\frac{(1-\mu_j)}{s}z_j. \quad (3.1.3)$$

Since $Q = conv\{f_1, f_2, \ldots, f_s\}$, each of z_j is a convex combination of extreme points $f_1, f_2, \ldots, f_s$. Symbolically, there is such row stochastic matrix $\{v_{ji}\}_{j,i=\overline{1;s}}$ that $z_j = \sum_{i=1}^{s} v_{ji} f_i$ for $j = \overline{1;s}$. After replacing z_j in (3.1.3) by the its representations via extreme points we have

$$x = \sum_{j=1}^{s}\left(\frac{\mu_j}{s} + \sum_{k=1}^{s}\frac{(1-\mu_k)}{s}v_{kj}\right)f_j. \quad (3.1.4)$$

Here $\frac{\mu_j}{s} + \sum_{k=1}^{s}\frac{(1-\mu_k)}{s}v_{kj} \geq \frac{\mu_j}{s} > 0$, so (3.1.4) is the desired convex representation for x.

Now we prove remained part of the theorem, namely x is described as a convex combination of all extreme points as $x = \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_s f_s$ with $\lambda_j > 0$ for $j = \overline{1;s}$ then $x \in ri(Q)$. We do this task using Lemma 3.3. Let $y \in Q / \{x\}$, then $\exists \sigma_1, \ldots, \sigma_s \in [0;\, 1)$ with $\sum_{i=1}^{s}\sigma_s = 1$ which $y = \sum_{i=1}^{s}\sigma_i f_i$. We get $\varepsilon := \frac{\min\{\lambda_1, \ldots, \lambda_s\}}{2\cdot\max\{\sigma_1, \ldots, \sigma_s\}} > 0$ hence $\lambda_j - \sigma_j \cdot \varepsilon \geq \sigma_j \cdot \varepsilon > 0$ for $j = \overline{1;s}$. Clearly, $\varepsilon < 1$ hence all $\delta_j := \frac{\lambda_j - \sigma_j \cdot \varepsilon}{1-\varepsilon}$ is positive and $\delta_1 + \ldots + \delta_s = 1$. Let us consider $z = \delta_1 f_1 + \ldots + \delta_s f_s$. Then $z \in Q$ by the its representation via $f_1, f_2, \ldots, f_s$ and we have

$$(1-\varepsilon)z + \varepsilon y = \sum_{j=1}^{s}(\lambda_j - \sigma_j \cdot \varepsilon) f_j + \varepsilon \cdot \sum_{j=1}^{s}\sigma_j f_j = \sum$$

Hence we conclude $x \in ri(Q)$ by the second assertion of Lemma 3.3. □

3.2. Description of the set of strictly regular quadratic bistochastic operators. In this subsection we give the main results of the paper. The following theoremdescribes the nature of the connection of dynamical systems of convex combination with dynamical systems of the operators which attend in that combination and it is the main theorem of the paper.

THEOREM 3.2. *Let $V_1, V_2, \ldots, V_t$ be $m-$ dimensional bistochastic operators and $\lambda_1, \lambda_2, \ldots, \lambda_t$ be positive numbers with $\sum_{i=1}^{t}\lambda_i = 1$. Then the following holds:*

i) $$Fix(\sum_{i=1}^{t}\lambda_i V_i) = \bigcap_{i=1}^{t} Fix(V_i);$$

ii) $$Per_p(\sum_{i=1}^{t}\lambda_i V_i) \subset \bigcap_{i=1}^{t} Per_p(V_i),$$ *for* $\forall p \in N$, *where* $Fix(V)$ *and* $Per_p(V)$ *denote the set of fixed points of* V *and* $p-$ *periodic points of* V *with prime period* p, *respectively.*

iii) Generally, reverse of the inclusion relation in the second statement does not hold.

PROOF. Without lost of generality we can suppose $t = 2$, because proof of the theorem in cases whose value of t is largercan be easily implied (via mathematical induction principle) by this simple case.

$i)$ $Fix(\lambda_1 V_1 + \lambda_2 V_2) \supset Fix(V_1) \bigcap Fix(V_2)$ is obvious, therefore, showing $Fix(\lambda_1 V_1 + \lambda_2 V_2) \subset Fix(V_1) \bigcap Fix(V_2)$ is sufficient. Let $x \in Fix(\lambda_1 V_1 + \lambda_2 V_2)$ that is $\lambda_1 V_1(x) + \lambda_2 V_2(x) = x$ (**). Then by the bistochasticity of V1 and V2, we have $V_1(x),\ V_2(x) \in \Pi_x$. But x is extreme point of the convex set Π_x by the Proposition2.1, then from the (**) and definition of extreme point we have $V_1(x) = V_2(x) = x$.

$ii)$ Let $x_0 \in Per_p(V)$ and the periodic orbit of x_0 is denoted with $x_i := V^i(x_0)$, herewith $y_i = V_1(x_{i-1})$, $z_i = V_2(x_{i-1})$ for $i = \overline{1; p}$, where $V := \lambda_1 V_1 + \lambda_2 V_2$. Bistochasticity of V, V_1, V_2 implies $x_i, y_i, z_i \in \Pi_{x_{i-1}}$ and $\Pi_{x_0} \subset \Pi_{x_1} \subset \ldots \subset \Pi_{x_p} = \Pi_{x_0}$ (due to $x_p = x_0$) by the determining of these sets. Hence, these sets are equal to each other. Proposition 2.1 implies that each of x_i, $i = \overline{1; p}$ is extreme point of $\Pi_{x_i} = \Pi_{x_0}$. By the definition of extreme point and according to equality $\lambda_1 y_i + \lambda_2 z_i = x_i \in \Pi_{x_0}$, we get $y_i = z_i = x_i$, for $i = \overline{1; p}$. Thence, trajectory of x_0 with respect to V, V_1 and V_2 is the same and every of them are $p-$ periodic, thus $x_0 \in Per_p(V_1) \bigcap Per_p(V_2)$.

$iii)$ We get linear bistochastic operators on S^2 which are given by their matrices in the standard basis as

$$P_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

It is easily checked that these operators are an example for not holding the reverse relation to the inclusion in the second statement. □

REMARK 3.1. It is worth mentioned that in the proof of the above theorem we do not use from quadraticity of the considered operator. Therefore, in the statement of this theorem we claim only bistochasticity of the operator.

COROLLARY 3.2. *Let V_1 be a strictly regular QBO and V_2 be a QBO, then $V_\lambda = \lambda V_1 + (1-\lambda)V_2$ is a strictly regular QBO for $\forall \lambda \in (0;\ 1)$. In particular, the set of strictly regular QBOs is convex.*

PROOF. According to Proposition 2.2, $Fix(V_1) = \left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)$ and $Per_p(V_1) = \varnothing$ for any $p \geq 2$. Then by the Theorem 3.2 we obtain $Fix(V_\lambda) = \left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)$ and $Per_p(V_\lambda) = \varnothing$ $p \geq 2$. Hence we again apply Proposition 2.2 and have strictly regularity of V_λ. □

As the mentioned above the set of m− dimensional quadratic bistochastic operators, B_m is convex, compact (closed and bounded) set. Hence by the Krein-Milmann theorem we have

$$\mathrm{B}_m = conv(Extr(\mathrm{B}_m)). \quad (3.2.1)$$

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Similarly, the set of linear bistochastic operators (Birkhoff polytope) is also convex, compact set and celebrated Birkhoff-von Neumann theorem states that extreme points of this set are finite and that are coordinate permutation operators. Obviously, a linear bistochastic operator is also QBO. Now we show that coordinate permutation operators are also extreme points of larger set B_m.

LEMMA 3.4. *Let* P *be a coordinate permutation operator, then* $P \in Extr(\mathrm{B}_m)$

PROOF. Firstly, we show that identical operator is extreme QBO. Bistochasticity of identical operator is obvious and let $\lambda V_1 + (1-\lambda)V_2 = id$ for some $\lambda \in (0;\ 1)$, $V_1, V_2 \in \mathrm{B}_m$. Then $\lambda V_1(x) + (1-\lambda)V_2(x) = x$ and $V_1(x),\ V_2(x) \in \Pi_x$ holds for $\forall x \in S^{m-1}$. x is a extreme point of Π_x, thus $V_1(x) = V_2(x) = x$ by the definition of extreme point. According to arbitrarily choosing of x, we have $V_1 = V_2 = id$.

Let P be a coordinate permutation operator, hence it is invertible and its inverse also coordinate permutation operator, so both of them is QBO. Let $\lambda V_1 + (1-\lambda)V_2 = P$ for some $\lambda \in (0;\ 1)$, $V_1, V_2 \in \mathrm{B}_m$. Thence we have $\lambda(P^{-1} \circ V_1) + (1-\lambda)(P^{-1} \circ V_2) = id$ (both $P^{-1} \circ V_1$ and $P^{-1} \circ V_2$ is QBO according to the second assertion of Theorem 2.1) and by the extremity of identical operator we obtain $P^{-1} \circ V_1 = P^{-1} \circ V_2 = id$. Hence $V_1 = V_2 = P$. □

We denote the group of $m-$dimensional coordinate permutation operators by P_m. Clearly, $|\mathrm{P}_m| = m!$. We number the elements of P_m with P_j, $j = \overline{1; m!}$. Lemma 3.4 asserts that $\mathrm{P}_m \subset Extr(\mathrm{B}_m)$ and the third assertion of Theorem 2.1 states that extreme points of B_m is finite. Let $s = |Extr(\mathrm{B}_m) \setminus \mathrm{P}_m|$ and $\{V_1, V_2, ..., V_s\} = Extr(\mathrm{B}_m) \setminus \mathrm{P}_m$. Thus $Extr(\mathrm{B}_m) = \mathrm{P}_m \bigcup \{V_1, V_2, ..., V_s\}$ and according to (3.2.1) relation we have $\mathrm{B}_m = conv(P_1, ..., P_{m!}, V_1, ..., V_s)$.

THEOREM 3.3. *Any operator in* $ri(\mathrm{B}_m)$ *is strictly regular.*

PROOF. Let $V \in ri(\mathrm{B}_m)$ be an operator in the relatively interior of B_m then it can be expressed as

$$V = \sum_{j=1}^{s} \lambda_j V_j + \sum_{j=1}^{m!} \lambda_{j+s} P_j$$

by the Theorem 3.1 where $\sum_{j=1}^{s+m!} \lambda_j = 1$ and $\lambda_j > 0$ for $j = \overline{1; (s+m!)}$. Then the first assertion of Theorem 3.2 implies

$$Fix(V) = \bigcap_{j=1}^{s} Fix(V_j) \cap \bigcap_{j=1}^{m!} Fix(P_j) \subset \bigcap_{j=1}^{m!} Fix(P_j) = \left(\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m}\right) \quad (3.2.2)$$

and according to the second assertion of Theorem 3.2 we obtain

$$Per_p(V) \subset \bigcap_{j=1}^{s} Per_p(V_j) \cap \bigcap_{j=1}^{m!} Per_p(P_j) \subset \bigcap_{j=1}^{m!} Per_p(P_j) = \varnothing \quad (3.2.3)$$

for a natural number $p \geq 2$.

According to the Proposition 2.2, the relations (3.2.2) and (3.2.3) implies regularity of V. □

REMARK 3.2. We note that Theorem 3.3 is proved via geometrical principles and the proof bases on the Theorem 3.2. This theorem can be also followed by the main theorem of **[6]** (Theorem 3.1 in that work).

COROLLARY 3.3. The set of strictly regular QBOs is dense in B_m.

PROOF. According to Corollary3.1 we have $\overline{ri(\mathrm{B}_m)} = \mathrm{B}_m$. □

4. Examples

In this section we give concrete examples to the strictly regular QBO's in any dimensions with the following theorem.

THEOREM 4.1. *Let* $A = \{a_{ij}\}_{i,j=1}^{m}$ *be a bistochastic matrix, then the following assertions hold:*

i) $$V_A(x)_k = \sum_{i=1}^{m} a_{ki} x_i^2 + x_k(1 - x_k)$$

$k = \overline{1;m}$ *is a bistochastic operator.*

ii) $$V_A(x)_k = \frac{1}{m}\sum_{i=1}^{m} x_i^2 + x_k(1 - x_k)$$

$k = \overline{1;m}$ *is a strictly regular QBO.*

PROOF. i) We make notation $x' := V_A(x)$. Then $x' \prec x$ is equivalent to $\forall k \in N_m = \{1, 2, ..., m\}$ and $\forall \{i_1, i_2, ..., i_k\} \subset N_m$ ($|\{i_1, i_2, ..., i_k\}| = k$) $\sum_{l=1}^{k} x'_{i_s} \le \sum_{j=1}^{k} x_{[j]}$.

Therefore, we will prove second equivalent assertion. Let $x_\downarrow = (x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(m)})$, namely $\pi \in S_m$ is suitable to permutation of the coordinates of x in non-increasing order, where S_m is the permutation group of N_m. Firstly, we will show $\sum_{j=1}^{m}\left(\sum_{s=1}^{k} a_{i_s j}\right) x_j^2 \le \sum_{j=1}^{k} x_{[j]}^2$ (*). Indeed, bistochasticity of A implies that $\sum_{s=1}^{k} a_{i_s t} \le 1$ ($\forall t \in N_m$) and $\sum_{j=1}^{m}\left(\sum_{s=1}^{k} a_{i_s j}\right) = \sum_{j=1}^{m}\left(\sum_{s=1}^{k} a_{i_s \pi(j)}\right) = k$. The last equality implies

$$\sum_{j=k+1}^{m}\sum_{s=1}^{k} a_{i_s\pi(j)} = k - \sum_{j=1}^{k}\sum_{s=1}^{k} a_{i_s\pi(j)} = \sum_{j=1}^{k}\left(1 - \sum_{s=1}^{k} a_{i_s\pi(j)}\right) \quad (4.0.1).$$

Hence and according to $x_{\pi(m)} \le x_{\pi(m-1)} \le ... \le x_{\pi(1)}$ we have

$$\sum_{j=k+1}^{m}\left(\sum_{s=1}^{k} a_{i_s\pi(j)}\right) x_{\pi(j)}^2 \le \sum_{j=1}^{k}\left(1 - \sum_{s=1}^{k} a_{i_s\pi(j)}\right) x_{\pi(j)}^2$$
(4.0.2).

(4.0.2) implies $\sum_{j=1}^{m}\left(\sum_{s=1}^{k} a_{i_s j}\right) x_j^2 = \sum_{j=1}^{m}\left(\sum_{s=1}^{k} a_{i_s\pi(j)}\right) x_{\pi(j)}^2 \le \sum_{j=1}^{k} x_{\pi(j)}^2$. We denote with $x_{[i_1]}, x_{[i_2]}, ..., x_{[i_k]}$ the non-increasing rearrangement of $x_{i_1}, x_{i_2}, ..., x_{i_k}$, then

$$\sum_{s=1}^{k} x'_{i_s} = \sum_{j=1}^{m}\left(\sum_{s=1}^{k} a_{i_s j}\right) x_j^2 + \sum_{s=1}^{k} x_{i_s}(1 - x_{i_s}) = \sum_{j=1}^{m}\left(\sum_{s=1}^{k} a_{i_s j}\right) x_j^2 +$$
$$+ \sum_{s=1}^{k}(x_{[i_s]} - x_{[i_s]}^2) \le \sum_{s=1}^{k} x_{[s]}^2 + \sum_{s=1}^{k} x_{[i_s]}(1 - x_{[i_s]}) = \sum_{s=1}^{k}(x_{[s]}^2 + x_{[i_s]}(1 - x_{[i_s]})) =_{\text{by}}$$
$$= \sum_{s=1}^{k}(x_{[s]} + (x_{[s]} - x_{[i_s]})(x_{[s]} + x_{[i_s]} - 1)) \le \sum_{s=1}^{k} x_{[s]}$$

by the $x_{[s]} \ge x_{[i_s]}$ and $x \in S^{m-1}$.

ii) Let $C_\sigma := \{x \in S^{m-1} : x_\downarrow = (x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(m)})\}$ where $\sigma \in S_m$ is a permutation of N_m. We prove each of C_σ is invariant w.r.t. V. Obviously, $V((x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(m)})) = (x'_{\sigma(1)}, x'_{\sigma(2)}, ..., x'_{\sigma(m)})$ for $\forall \sigma \in S_m$, where $(x'_1, x'_2, ..., x'_m) = V((x_1, x_2, ..., x_m))$. Consequently, we can suppose $x \in C_{id}$, i.e. $x_1 \ge x_2 \ge ... \ge x_m$. We take $\forall i, j \in N_m$ $i < j$. Then $x'_i - x'_j = x_i(1 - x_i) - x_j(1 - x_j) = (x_i - x_j)(1 - x_i - x_j) \ge 0$. Hence $x' \in C_{id}$. Thus we show $V : C_\sigma \to C_\sigma$ for $\forall \sigma \in S_m$ and V are bistochastic. Hence any trajectory of V converges some point in

$Fix(V)$ by the its bistochasticity .Let $p \in Fix(V)$, then $V(p) = p$ implies that $p_i^2 = \frac{1}{m}\sum_{i=1}^{m} p_j^2$, $i = \overline{1;m}$. By the last equalities, we have $p = \left(\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m}\right)$, thus $Fix(V) = \left\{\left(\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m}\right)\right\}$. Thence any trajectory of V converges to the unique fixed point $\left(\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m}\right)$. $\square$

REMARK 4.1. We note that strictly regularity of the operator in the above theorem is proved by using the fact that it is monotonic (order-preserving map). It is worth mentioned that the second statement of the theorem can be also proven via applying the main theorem of **[6]** (Theorem 3.1 in that paper) and this method of proving is completely different from ours.

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