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SUB-MARKOV RANDOM WALK FOR IMAGE SEGMENTATION WITH PRIOR LABELS

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ABSTRACT

Segmentation is the first step in object identification in any image. It can also be used to compress different areas or different segments of an image, at different compression qualities. So, for the segmentation of an image we have developed a novel technique known as sub-Markov random walk (subRW) algorithm with label prior for seeded image segmentation. This is similar to the traditional random walk with auxiliary nodes added in it. Under this auxiliary nodes consideration we have given uniqueness of proposed work than existing systems. Our method is more efficient compared to previous work. The uniqueness will be nothing but adding or changing the auxiliary nodes in segmentation algorithm. We face segmentation problem in existing system if the image is having very thin and elongated parts. To solve this type of problem we implemented proposed work. Matlab simulation results proved thatour proposed subRW is giving better results compare to all other existing RW algorithms.

KEYWORDS: Seeded image segmentation, prior labels, sub-markov random walk, optimization.

I. INTRODUCTION

Image segmentation will play a major role in image processing. So practically image segmentation algorithm must provide four qualities. Those are

- 1) fast computation
- 2) fast editing
- 3) producing an arbitrary segmentation with enough interaction
- 4) intuitive segmentations. Mostly used algorithms areRW[1],LRW[3],RWR[2],PAR W[4]. By using the random walk algorithm we can get

all the above desired qualities [11]. By using

sparse, symmetric positive definite system of linear equations. The random walk algorithm may give good results by taking the solution of previous one as the initialization of an iterative matrix solver. The algorithm formation on a graph allows the application of the algorithm to surface meshes or space variant images [12], [13]. In K-way image segmentation user defined seeds are given to indicate the regions of the image belonging to K objects. Every seed indicates the location with user defined

some methods we can solve the problem of



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label. Random walker first consider the seed points which exactly equals the solution to the Dirichlet problem [14] and the seed point is fixed to unity remaining are set to zero. In this paper we advocate a sub Markov random walk (subRW). It can have four RW-based algorithms: RW[1], RWR[2], LRW[3] and PARW[4]. To solve the twig problem we add prior labels. In the residences of sub- Markov random walk, first step is to construct a subRW framework for photograph segmentation. First it will go away a graph G from a node i having the probability of ci and then it will pass to the alternative adjacent nodes in G having the possibility of 1-ci. This random walk is changed to a Markov transition probability $(\Sigma q(i, j) = 1)$ in an expanded graphGe. This Ge is built through including auxiliary staying nodes connected with seeds and unseeded nodes into graph G are related with auxiliary killing nodes.

II. LITERATURE SURVEY

For the segmentation of medical images so many approaches have been proposed. These approaches generally grouped into two main categories. Those semiautomatic and fully automatic method. Random walker algorithm will come under as semi-automatic method. It is proposed by grady and other methods based on graph cuts [15] for the segmentation of regions user should provide seed points. By using all these methods the final segmentation and results are obtained. When a large batch of images is there for segmentation these methods are practically not used. In automatic methods, it does not require any manual interaction. So many conventional methods have been proposed in the past years. Those methods include [6],[5],[7],[8] . In [6] random walk is extended for disconnected objects with out labeling.[5] provides shortest path algorithms. Adding watershed segmentation to this framework[7] makes theoretical analysis.Leo Grady, Member,[1].proposed Random walks for image segmentation. First the graph is partitioned into nodes (seed point). Given a random walker starting at any location, what is the probability that it first reaches each of the K seed points.B. Ham, D. Min, and K. Sohn[2] proposed random walk with restarting probabilities. In RWR algorithm a random walker walk to other adjacent nodes with probability (1-c)and returns to the starting node with a probability c. J. Shen, Y. Du, W. Wang, and X. Li,[3] proposed lazy random walks. According to this algorithm random walkerstays at the present node with a probability $1 - \alpha$ and walks along the edges connected with the presentnode with probability α . Mostly we use edge weight computation method in graph-based image segmentation approaches to represent the image intensity changes.X.-M. Wu, Z. Li, A. M.So, J.

Wright, and S.-F.Chang[4] proposed Learning with partially absorbing random walks. According to this algorithm, a random walker is absorbed at current node I with a probability αi and follows a random edge out of it with probability $1 - \alpha i$. under as semi-automatic method. It is proposed by grady and other methods based on graph cuts[15] for the segmentation of regions user should provide seed points. By using all these methods the final segmentation and results are obtained. When a large batch of



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III. A UNIFYING VIEW OF SUBRW

By using the sub-Markov transition probability (subRW) random walk set of rules is proposed for the interactive multicategorized photo segmentation and the analysis become done between the preceding and proposed popular RW algorithms which include RW[1], RWR[2], LRW[3], and PARW[4]. In our technique we need to indicate multi label seeds on foreground and background image. random walker stays at the present node with a probability $1 - \alpha$ and walks along the edges connected with the present node with probability α . Mostly we use edge weight computation method in graph-based image segmentation approaches to represent the image intensity changes. X.- M. Wu, Z. Li, A. M.So, J. Wright, and S.- F. Chang [4] proposed Learning with partially absorbing random walks. According to this algorithm, a random walker is absorbed at current node i with a probability αi and follows a random edge out of it with probability $1 - \alpha i$.

A. The Sub-Markov Random

In the random-walkalgorithm,let weighted graph be G, including labeled nodes VM, and unlabeled nodes VU, so as VUUVM =V. where V gives total nodes.Now theseeded segmentation subRW algorithm is used. To get a graph G and Expanded graph Ge kinds of auxiliary nodes are defined.

B. The Optimization of SubRW:

SubRW set of rules mainly depends on random walks, it is able to be taken as a well known optimization problem. Bythe use of the optimization, we can hire the general vision utility.

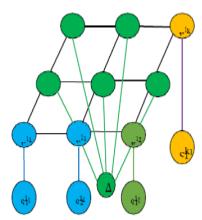


Figure.1.

SubRW nodes graph. Original nodes in V denoted in ellipse nodes and newly added auxiliary nodes are denoted in circles. The unseeded nodes are green color nodes and the remaining nodes are seeded nodes.



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C. Relations with Other Well-Known RW Algorithms:

The proposed results are compared with the conventional popular algorithms to see the performance. RW[1], RWR[2], LRW[3], and PAPW[4].

1) Relations with RW:

For the segmentation, Grady [1] proposed random walk algorithm with a Markov transition probability. This can betaken as a special case of subRW. In this algorithm it places a random walker at each unlabeled node then calculates all the labeled nodes. This calculation is not practical.

2) Relations with RWR:

Kim et al.[2] proposed random walk with restarting probability. In this for each node assigned a steady state probability and it develops a seeds.

3) Relations with LRW:

In [3] super pixel segmentation has been done by using the lazy random-walk algorithm and multi labeled segmentation problem is removed by using this algorithm. In this random walker will stay in the present position and walk out with an arbitrary edge.

4) Relations with PARW:

Ranking, clustering, and classification were done by using the partially absorbing random walks (PARWs)[4]. The PARW starts at current node i with the probability β i and walk out of it having a random edge with probability 1- β i. e., wihk α uk.

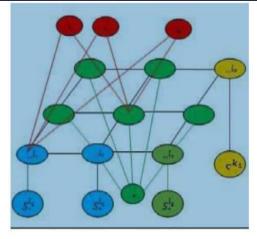


Figure. 2.

SubRW prior nodes of a node graph. Red circles-prior nodes (these should connect with all original ellipsenodes) for each prior node only two edges, one for seeded node and another for unseeded node. Red circles is the only difference between fig.1 and fig.2 In this paper, we set weight wihk as: k wihk = $(\Box - ci) \times ui$ (1) where λ is a regularization parameter, which measures the importance of the prior distribution. Then the transition probability is expressed As

$$\bar{q}(i,j) = \begin{cases} c_i \text{ifi} \in V, \text{ifj} \in \{\Delta\} \cup S_M \\ (1 - c_i) \frac{i}{\text{i}}, \text{ ificV, } j = h_k \\ w_{ij}, \text{ ificV} \cup S_M \cup H_M \end{cases}$$

$$(1 - c_i) \frac{1}{\text{d}_{i+} \times \mathbf{g}_i}, \text{ ifficV} \cup S_M \cup H_M$$

$$(2)$$

$$1 \qquad \qquad \text{ifi} = jc \{\Delta\}$$

$$0 \qquad \qquad \text{otherwise}$$

Hence, a transition probability q^- on a graph with prior \overline{G} , the probability rim the random walker from one node position reaches to the m-th staying node S1k with label 1 or prior node h, is expressed as follows

$$\Gamma_{\text{im}}^{-l_{k}} = (1 - c) \sum_{i} \frac{w_{ij} r_{iml_{k}}}{d_{i} + \lambda_{g_{i}}} + (1 - c) \frac{\lambda u^{k}}{d_{i} + \lambda_{g_{i}}} + c b^{1_{k}} (3)$$

The present prior node hk is seen as a new staying node with label lk. So, the reaching



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probability of hk is to be considered by the vector as

$$\begin{split} &\bar{r}_{m}^{l_{k}} = (I - D_{c}) \, \bar{p} \bar{r}_{1}^{l_{k}} + (I - D_{c}) \, \bar{u}^{k} + D_{c} b_{\, im}^{\, l_{k}} \\ &= (I - (I - D_{c}) \, \bar{p})^{-1} ((I - D_{c}) \, \bar{u}^{k} + D_{c} b_{\, im}^{\, l_{k}}) \\ &= \bar{E}^{-1} ((I - D_{c}) \bar{u}^{k} + D_{c} b_{\, l_{k}}) \\ &= \bar{m} \end{split} \tag{4}$$

The modified vector notation rm can be formulated as

$$\bar{r}^{l_k} = \frac{1}{Z_k} \bar{E}^{-1} \left((I - D) \bar{u}^k + \frac{1}{M_k} D b^{l_k} \right)$$

The final labeling (segmentation) result with a label prior is obtained as follows:

$$\bar{\mathbf{p}}_i = \arg \max_{1_k} \bar{\mathbf{r}}_{l^k}$$
 (6)

Where R^- represents the final node for each pixel of an image.B. The Optimization ExplanationSimilar to the subRW, we can also give the optimization explanation for the subRW with label prior. Suppose $\forall i$, \Box < ci < 1, then the objective function is as follows:

$$\begin{split} \bar{0}^{l_{k}} &= \frac{1}{2} \sum_{i=1}^{N} \sum_{i=1}^{N} w_{i,j} (r^{-l_{k}} - r^{-l_{k}})^{2} \\ &+ \frac{1}{2} \sum_{i=1}^{N} \frac{(d_{i} + \lambda_{g_{i}}) c_{i}}{(1 - c_{i})} W_{i,j} (r^{-l_{k}} - r^{-l_{k}})^{2} \\ &+ \frac{1}{2} \sum_{i=1}^{N} \frac{(d_{i} + \lambda_{g_{i}}) c_{i}}{(1 - c_{i})} W_{i,j} (r^{-l_{k}} - r^{-l_{k}})^{2} \\ &+ \frac{1}{2} \sum_{i=1}^{N} \frac{k}{u_{i}} (r^{-l_{k}} - 1) + \frac{1}{2} \sum_{t=1,t^{*}k}^{N} \sum_{i=1}^{N} \lambda_{i} \frac{t^{-l_{k}}^{2}}{u_{i}} (7) \end{split}$$

Algorithm 1 segmentation by subRW with label prior Input: an image V(ri) and K kinds of user scribbles VM ={V11,V12,....V1k}, the parameters Dc = diag(c1, c2, ... cN),\u00eds. y:

Output: The segmentation result R⁻ for each pixel:

- 1: Obtain the indicating vectors b1k,k=1: K of scribbles:
- 2: Define an adjacency matrix $W=[wij]N\times N$ with neighbors graph structure by (1);
- 3: Generate the GMMs with five components from all scribbles and then get the probability

density uk:

- 4: Scale probability: uk
- 5: if y = 1then $\leftarrow \max \square \square \square \square + \log \square uk)$, 10 -10):
- 6: obtain coarse segmentation result CRi = arg mas ui;

k

- 7: get the candidate vector crk from CRi;
- 8: reset the prior vector $uk \leftarrow uk \odot crk$;
- 91:0 :e n Sde itfg i \leftarrow gi , where ui +mas ui ; k k k

t*k

11: Compute the transition probability matrix P⁻ and the vector u⁻ k by (37) and (38);

1

12: solve linear equations: $E^{-} \overline{r}1k = (I -Dc)u^{-}k + Dcb1k;$

Mk

Zk

14: Obtain segmentation resultR⁻ = argmaxlk r⁻ s

1k; The vector formation of above equation is given as



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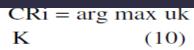
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$$\begin{split} \bar{0}^{lk} &= \overset{\cdot}{\Gamma}^{k,\Gamma}(D-W)r^{lk}_{m} \\ &+ \overset{\cdot}{L}(r^{-l_{k}^{2}} - b^{l_{k}})^{T}(I-D)^{-1}(D+\lambda D)D_{m}(r^{-l_{k}} - b^{l_{k}}) \\ &+ \overset{\cdot}{L}(r^{-l_{k}^{2}} - b^{l_{k}})^{T}(I-D)^{-1}(D+\lambda D)D_{m}(r^{-l_{k}} - b^{l_{k}}) \\ &+ \overset{\cdot}{L}(r_{m} - e) D_{u}(r_{m} - e) + \overset{\cdot}{L}(r^{-l_{k}^{2}} - b^{l_{k}^{2}}) \\ &+ \overset{\cdot}{L}(r_{m} - e) D_{u}(r_{m} - e) + \overset{\cdot}{L}(r^{-l_{k}^{2}} - b^{-l_{k}^{2}}) D_{u}r_{m} \\ &+ \overset{\cdot}{L}(r_{m} - e) D_{u}(r_{m} - e) + \overset{\cdot}{L}(r^{-l_{k}^{2}} - b^{-l_{k}^{2}}) D_{u}r_{m} \\ &+ \overset{\cdot}{L}(r_{m} - e) D_{u}(r_{m} - e) + \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} \\ &+ \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} + \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} \\ &+ \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} + \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} \\ &+ \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} + \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} \\ &+ \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} + \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} \\ &+ \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} + \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} \\ &+ \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} + \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} \\ &+ \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} + \overset{\cdot}{L}(r_{m} - e) D_{u}r_{m} \\ &+ \overset{\cdot}{L}(r_{m} - e)$$

From equation (8), we can find that it comprises of three segments. The initial two segments are the smooth term andthe unary term, which are like the parts in (7). The last component relates to the label prior. By limiting this component,—1kthe achieving likelihood rm will be reliable with the label prior.

C. Noise Reduction

Adding the label prior to each node of a pixel may produce some noise. This noise may produce some distortions. To avoid this one of the solution is to decrease parameter β . If the value of β is too small, the twig part may get lost. Weneed to some other strategies to reduce noise like combining label prior value for each node. Coarse segmentation is achieved to avoid lose of twig part. Coarse segmentation is represented as



As we know there will be much noise in the coarse segmentation, it mostly does not connect with the main part of an object. So, we will select the linked regions with seeds from the coarse segmentation as the candidate areas. Next weare able to every label earlier into these candidate regions, which facilitates to keep the earlier records of twig element as well as noise gets eliminated. Furthermore, the candidate regions are dilated to add earlier information into the boundary areas. Since the assessment of the prior facts near limitations is high, it'll assist to discover correct barriers.

V. SIMULATION RESULTS



Figure 1: (a) Input image 1



(b) Input image 2



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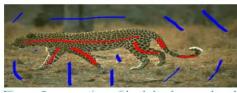
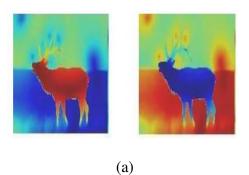
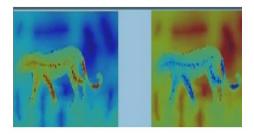


Figure 2: separation of both background and foreground regions(scribbled imagesfor input 1 and input 2)





(b)Figure.3. True color images with range of color maps



Figure.4. segmentation results

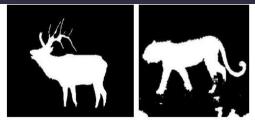


Figure.5: extraction of binary image

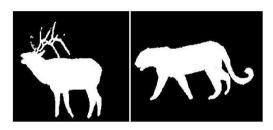
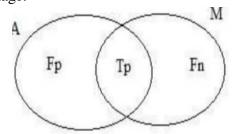


Figure.6: Manually segmented images **VI. EVALUATION METRIC**

The results of evaluation of image is obtained by the proposed method and is compared with the manually segmented image.. Let us represent 'M' be the manual segmented image and 'A' be the segmented image.



The Similarity Index (SI), Correct Detection Ratio (CDR), Under Segmentation Error (USE) and Over Segmentation Error (OSE) are used for efficient evaluation. SI is a measure which offers true segmented region relative to the total segmented region. CDR indicates the degree of trueness of the actual image. USE is the ratio of the number of pixels falsely identified as segmented image by the proposed method to that of manual segmented image. OSE is the ratio of



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number of pixels falsely identified as unsegmented image by the proposed method to that of manual segmented image. We define Total Segmentation Error (TSE) as the sum of USE and OSE. The evaluation metrics SI, CDR, USE and OSE are obtained by equations

$$SI = \frac{2Tp}{2Tp + Fp + Fn} \times 100 \%$$

$$CDR = \frac{Tp}{Tp + Fn} \times 100 \%$$

$$USE = \frac{2Fn}{Tp + Fn} \times 100 \%$$

$$OSE = \frac{Tp}{Tp + Fn} \times 100 \%$$

$$TSE = USE + OSE$$

| | Input 1 | Input 2 |
|-------------------------------|---------|---------|
| Similarity index(SI) | 0.8837 | 0.9285 |
| Correct detection ratio(CDR | 0.8835 | 0.8917 |
| Under segmentation error(USE) | 0.1160 | 0.0289 |
| Over segmentation error(OSE) | 0.1165 | 0.1083 |
| Total segmentation error(TSE) | 0.2325 | 0.1372 |

Table: Evaluation metric

VI. CONCLUSION

We have conferred a completely unique framework supporting the subMarkov random walk for interactive seeded image segmentation in this work. This framework are often explained as a traditional random walker that walks on the graph by adding some new auxiliary nodes, that makes our framework simply interpreted and a lot of versatile. Below this framework, we unify the well-known RW-based algorithms that satisfy the subMarkov property and build bridges to create it simple to remodel the findings between them. What is more, we've got designed a novel subRW with label before solve the twigs segmentation problems by adding previous nodes into our framework. The experimental results have shown that our algorithmic outperforms the progressive RW-based algorithms. This also proves that it's practicable to style a replacement subRW algorithmic program by adding new auxiliary nodes into our framework. We can extend our algorithmic program to a lot of applications, such as center line detection at 3D medical pictures and classification. Finally some of the parameters of an algorithm have been changed to reduce the noise of an image as well as to extract the particular region of an image with perfect edges and flexibility.

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